

ON LOGARITHMIC SOBOLEV INEQUALITY FOR THE NONCOMMUTATIVE TWO TORUS

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ABSTRACT. An analogue of Gross' logarithmic Sobolev inequality for a class of elements of noncommutative two tori is proved.

1. INTRODUCTION

The subject of logarithmic Sobolev inequality has its roots in the paper of E. Nelson [9], where he proved the contractivity of the semi-group generated by the Gauss-Dirichlet form operator. Then, shortly after that L. Gross introduced logarithmic Sobolev inequalities in [4] and using them gave a different proof of the contractivity of the semi-group generated by Gauss-Dirichlet form operator.

Let ν be the Gauss measure on \mathbb{R}^n and

$$N : D(N) \subseteq L^2(\mathbb{R}^n, \nu) \longrightarrow L^2(\mathbb{R}^n, \nu),$$

be the Gauss-Dirichlet form operator defined by

$$\langle Nf, g \rangle = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x) d\nu(x)$$

where $\langle Nf, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)} d\nu(x)$, and ∇f is the weak gradient of f . Nelson showed that for $1 < q \leq p < \infty$, if

$$e^{-2t} \leq \frac{(q-1)}{(p-1)},$$

then

$$\|e^{-tN}\|_{q \rightarrow p} \leq 1,$$

where

$$\|e^{-tN}\|_{q \rightarrow p} = \sup \{ \|e^{-tN} f\|_p : f \in L^2(\nu) \cap L^q(\nu), \|f\|_q \leq 1 \}.$$

This means that for

$$t \geq \ln \sqrt{\frac{p-1}{q-1}},$$

e^{-tN} is a contraction from $L^q(\mathbb{R}^n, \nu)$ to $L^p(\mathbb{R}^n, \nu)$. He even proved more. Indeed, he showed that e^{-tN} is an unbounded operator from $L^q(\mathbb{R}^n, \nu)$ to $L^p(\mathbb{R}^n, \nu)$, if

$$t < \ln \sqrt{\frac{p-1}{q-1}}.$$

The classical Sobolev inequality states that for $f \in C_c^\infty(\mathbb{R}^n)$,

$$(1) \quad \|f\|_{L^q(\mathbb{R}^n, dx)} \leq C_{p,n} \|\nabla f\|_{L^p(\mathbb{R}^n, dx)}$$

where $1 \leq p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, dx is the Lebesgue measure and $C_{p,n}$ is a constant depending only on n and p . So (1) implies that if the gradient of the function f is in $L^p(\mathbb{R}^n, dx)$, then f must be in $L^q(\mathbb{R}^n, dx)$. These inequalities strongly depend on the dimension of \mathbb{R}^n .

In [4], Gross proved a logarithmic Sobolev inequality

$$(2) \quad \int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| d\nu(x) \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 d\nu(x) + \|f\|_2^2 \ln \|f\|_2$$

for $f \in L^2(\mathbb{R}^n, \nu)$, and showed that this inequality is equivalent to Nelson's result of contractivity that we just mentioned.

Unlike the classical Sobolev inequality, Gross' logarithmic Sobolev inequality is dimension independent. Using (2) and the Poincaré inequality we see that if the gradient of the function f is in $L^2(\mathbb{R}^n, \nu)$, then f is in the Orlicz space $L^2 \ln L(\nu)$. This somehow justifies the name logarithmic Sobolev inequality. Gross also derived in [4] a weaker version of (2), from Nirenberg's form of classical Sobolev inequality [10]. This version, not surprisingly depends on the dimension.

Since then people have given various proofs of logarithmic Sobolev inequalities by different methods: O. Rothaus [13] has proved it using Jensen's inequality and the positivity of the lowest eigenfunction for a Sturm-Liouville boundary value problem with Dirichlet boundary conditions. R. A. Adams and F. H. Clarke also have given a simple proof based on calculus of variations [1].

One can replace (\mathbb{R}^n, ν) with a probability space (X, μ) and the Gauss-Dirichlet form with a densely defined positive quadratic form on $L^2(X, \mu)$, say \mathcal{E} . Then we say the logarithmic Sobolev inequality holds for \mathcal{E} , if for $f \in \text{Dom}(\mathcal{E})$,

$$\int_X |f(x)|^2 \ln |f(x)| d\mu(x) \leq \mathcal{E}(f, f) + \|f\|_2^2 \ln \|f\|_2.$$

This way one can talk about logarithmic Sobolev inequalities on Riemannian manifolds [6].

F. Weissler has proved in [15] a logarithmic Sobolev inequality on the circle. Indeed, using Fourier series he has shown that for a positive function f in $L^2(\mathbb{T}, \mu)$, where \mathbb{T} is the unit circle and μ is the normalized Lebesgue measure, if

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

then

$$(3) \quad \int_{\mathbb{T}} f \log f d\mu \leq \sum_{n=-\infty}^{\infty} |n| |a_n|^2 + \|f\|_2^2 \log \|f\|_2.$$

Since

$$\sum_{n=-\infty}^{\infty} |n| |a_n|^2 \leq \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 = \|\nabla f\|_2^2 = \int_{\mathbb{T}} |\nabla f|^2 d\mu,$$

Weissler's result is even stronger than the original logarithmic Sobolev inequality

$$\int_{\mathbb{T}} f \log f d\mu \leq \int_{\mathbb{T}} |\nabla f|^2 d\mu + \|f\|_2^2 \log \|f\|_2.$$

There is a useful survey of related topics and applications of logarithmic Sobolev inequalities by Gross in [5]. One can also find more references therein.

Since the introduction of noncommutative geometry by Alain Connes in [2] (see also [3]), noncommutative tori have proved to be an invaluable tool to understand and test many aspects of noncommutative geometry that is not present in the commutative case. The results are simply too many to be cited here. The present paper should be seen as a step in understanding aspects of measure theory and analysis on noncommutative tori that have been largely untouched so far. The combinatorial challenges one faces in extending the logarithmic Sobolev inequality, at least in the form that we understand it, seemd to us as a very interesting problem by itself.

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The universal C^* -algebra generated by two unitaries U, V such that $UV = e^{2\pi i \theta} VU$, is called the irrational rotation algebra and is denoted by A_θ . It is a simple algebra and has a unique positive faithful normalized trace τ . A_θ is the noncommutative deformation of $C(\mathbb{T}^2)$, the algebra of continuous functions on the 2-torus. More details

about A_θ can be found in [3, 7, 11]. Let

$$A_\theta^\infty = \left\{ \sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} U^m V^n : a_{m,n} \text{ is rapidly decreasing} \right\}.$$

By rapidly decreasing we mean for all $k \in \mathbb{N}$,

$$\sup_{(m,n) \in \mathbb{Z}^2} (1 + m^2 + n^2)^k |a_{m,n}|^2 < \infty.$$

A_θ^∞ is a dense subalgebra of the irrational rotation algebra and it is the analogue of $C^\infty(\mathbb{T}^2)$, the algebra of smooth functions on the 2-torus. A_θ^∞ is called the noncommutative two torus.

Let $a = \sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} U^m V^n$ be in A_θ^∞ . Then

$$\begin{aligned} a^* &= \sum_{(m,n) \in \mathbb{Z}^2} \overline{a_{m,n}} V^{-n} U^{-m} \\ &= \sum_{(m,n) \in \mathbb{Z}^2} \overline{a_{m,n}} e^{-2\pi i m n \theta} U^{-m} V^{-n} \\ &= \sum_{(m,n) \in \mathbb{Z}^2} \overline{a_{-m,-n}} e^{-2\pi i m n \theta} U^m V^n. \end{aligned}$$

So if $a = a^*$, we have

$$(4) \quad a_{m,n} = \overline{a_{-m,-n}} e^{-2\pi i m n \theta}.$$

Moreover, if $b = \sum_{(m,n) \in \mathbb{Z}^2} b_{m,n} U^m V^n \in A_\theta^\infty$, we have

$$ab = \sum_{(m,n) \in \mathbb{Z}^2} c_{p,q} U^p V^q,$$

where

$$(5) \quad c_{p,q} = \sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} b_{p-m, q-n} e^{-2\pi i (p-m)n\theta}$$

The unique trace τ on A_θ which plays the role of integration in the noncommutative setting, extracts the constant term of the elements of A_θ^∞ , i.e. $\tau(a) = a_{0,0}$. This trace can be used to define an L^2 -norm on A_θ^∞ by

$$\|a\|_2^2 := \tau(a^* a).$$

Using (4) and (5) one can show $\|a\|_2^2 = \sum_{(m,n) \in \mathbb{Z}^2} |a_{m,n}|^2$. One can also

define Sobolev norms on A_θ^∞ . For more details see [12], where J. Rosenberg has developed the Sobolev theory on the noncommutative two torus.

In this paper we will use Weissler's method [15] to prove a logarithmic Sobolev inequality for a class of elements of the noncommutative two torus. In section 2 we will prove some lemmas that we will need later on. In section 3 we will first state our conjecture about a logarithmic Sobolev inequality on the noncommutative 2-torus and then we will prove that conjecture for a class of elements of the noncommutative 2-torus. This would be the main result of this paper. Although we have not been able to prove the logarithmic Sobolev inequality for arbitrary positive elements, we think the inequality must hold for those elements as well. In section 4 we will try to generalize the proof of the main result to prove the conjecture, but in the middle of the way we will see that we will face a problem. We hope that we can bypass this problem in a follow up paper.

2. PRELIMINARIES

In this section we will prove some technical lemmas that will be needed later on.

Lemma 2.1. *Let G be an analytic function in some complex neighborhood of the interval $[0, 1]$. Suppose all the coefficients in the power series expansion of G around $r = 0$ are nonnegative. Then $G(1) \geq 0$.*

Proof. First we show that we can find finitely many points $x_0 = 0 < x_1 < x_2 < \cdots < x_n = 1$ in $[0, 1]$ and finitely many discs $D_0, D_1, D_2, \dots, D_n$ such that x_i for $i = 0, 1, \dots, n$, is the center of D_i , G has a power series expansion around x_i on D_i and $x_i \in D_{i-1}$, for $i = 1, 2, \dots, n$. To show this let N be the open set in \mathbb{C} containing $[0, 1]$ on which G is analytic. Define

$$F : [0, 1] \longrightarrow \mathbb{R}^{>0}$$

by sending $r \mapsto \text{dist}(r, N^c)$. F is a continuous function on a compact set, so it attains its minimum. Let δ be the minimum of F and $x_0 = 0, x_1 = \frac{\delta}{2}, x_2 = \delta, x_3 = \frac{3\delta}{2}, \dots, x_{n-1} = \frac{(n-1)\delta}{2}, x_n = 1$, where $n = \lfloor \frac{2}{\delta} \rfloor + 1$. For $i = 0, 1, \dots, n$, let R_i be the radius of convergence of the power series expansion of G around x_i , and D_i be the disc centred at x_i with radius R_i . For $i = 0, 1, \dots, n$, we have $\frac{\delta}{2} < R_i$. So $x_i \in D_{i-1}$, for $i = 1, 2, \dots, n$.

Let

$$(6) \quad G(z) = \sum_{k=0}^{\infty} \frac{G^{(k)}(0)}{k!} z^k$$

be the power series expansion of G around 0 on D_0 . Since $x_1 \in D_0$, we plug x_1 in (6) for $k \geq 0$, $G^{(k)}(0) \geq 0$,

$$G(x_1) = \sum_{k=0}^{\infty} \frac{G^{(k)}(0)}{k!} x_1^k \geq 0$$

If we substitute x_1 into the derivative of (6), we will get

$$G^{(1)}(x_1) = \sum_{k=1}^{\infty} \frac{G^{(k)}(0)}{(k-1)!} x_1^{k-1}$$

which is non-negative by the same reason. Differentiating (6) repeatedly, we can show that the derivatives of G at x_1 which form the coefficients of the power series expansion of G around x_1 on D_1 are nonnegative.

Repeating this argument, we can show that all derivatives of G at each x_i and in particular at $x_n = 1$ are non-negative. So $G(1) \geq 0$. \square

We will need the following standard and elementary result of spectral theory in C^* -algebras.

Proposition 1. *Let A be a C^* -algebra, $x \in A$ and N an open subset of \mathbb{C} containing the spectrum of x , $\sigma(x)$. Then there exists $\delta > 0$, such that for $y \in A$, $\|y - x\| < \delta$ implies $\sigma(y) \subseteq N$.*

Proof. See [14] Theorem 10.20. \square

The following proposition will be needed in the proof of the main result of this paper.

Proposition 2. *Let $a = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} a_{m,n} U^m V^n$ be in A_θ^∞ , such that $a > 0$,*

$a_0 = 1$ and at most finitely many number of $a_{m,n}$'s are nonzero. For $r \in \mathbb{C}$, we put

$$x_r = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} a_{m,n} r^{(|m|+|n|)} U^m V^n$$

and $P_r(a) = 1 + x_r$. Then there is an open neighbourhood W of $[0, 1]$ in \mathbb{C} , such that for all r in W , $\log P_r(a)$ can be defined.

Proof. Since a is self-adjoint, using (4) for real r we have

$$a_{m,n} r^{(|m|+|n|)} = \overline{a_{-m,-n}} r^{(|-m|+|-n|)} e^{-2\pi i m n \theta}.$$

So $P_r(a)$ is self-adjoint for real r , and consequently the spectrum of $P_r(a)$ is real for real r . Now we show for $0 \leq r \leq 1$, $P_r(a)$ is a strictly positive element. Suppose for some $0 \leq r \leq 1$, $P_r(a)$ is not

strictly positive. Let $[t_0, t_1]$ be the smallest closed interval containing the spectrum of $P_r(a)$. We know that there exists a state ϕ of A_θ , such that $\phi(P_r(a)) = t_0 \leq 0$. Now let

$$\begin{aligned} M &= \{(m, n) \in \mathbb{Z}^2 : (m, n) \neq 0, a_{m,n} \neq 0\}, \\ M_1 &= \{(m, n) \in M : m \geq 0, n \geq 0\}, \\ M_2 &= \{(m, n) \in M : m > 0, n < 0\}. \end{aligned}$$

Then since a is self-adjoint, using (4) we have

$$\begin{aligned} (7) \quad a &= 1 + \sum_{(m,n) \in M} a_{m,n} U^m V^n \\ &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} U^m V^n + \sum_{(-m,-n) \in M_1 \cup M_2} a_{m,n} U^m V^n \\ &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} U^m V^n + \sum_{(m,n) \in M_1 \cup M_2} a_{-m,-n} U^{-m} V^{-n} \\ &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} U^m V^n + \sum_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}} e^{-2\pi i m n \theta} U^{-m} V^{-n} \\ &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} U^m V^n + \sum_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}} e^{-2\pi i m n \theta} e^{2\pi i m n \theta} V^{-n} U^{-m} \\ &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} U^m V^n + \sum_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}} V^{-n} U^{-m}. \end{aligned}$$

By the same reasoning we can show

$$\begin{aligned} P_r(a) &= 1 + \sum_{(m,n) \in M} a_{m,n} r^{(|m|+|n|)} U^m V^n \\ &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} r^{(|m|+|n|)} U^m V^n \\ &\quad + \sum_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}} r^{(|m|+|n|)} V^{-n} U^{-m}. \end{aligned}$$

Since a is strictly positive, using (7), we see that

$$\begin{aligned} \phi(a) &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} \phi(U^m V^n) \\ &\quad + \sum_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}} \phi(V^{-n} U^{-m}) > 0. \end{aligned}$$

Let $h_{m,n} = a_{m,n} \phi(U^m V^n)$. Then regarding the fact that

$$\phi(U^m V^n) = \overline{\phi(V^{-n} U^{-m})},$$

we have

$$(8) \quad \phi(a) = 1 + \sum_{(m,n) \in M_1 \cup M_2} (h_{m,n} + \overline{h_{m,n}}) > 0.$$

On the other hand,

$$\begin{aligned} \phi(P_r(a)) &= 1 + \sum_{(m,n) \in M_1 \cup M_2} a_{m,n} r^{(|m|+|n|)} \phi(U^m V^n) \\ &+ \sum_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}} r^{(|m|+|n|)} \phi(V^{-n} U^{-m}) = t_0 \leq 0. \end{aligned}$$

So

$$(9) \quad \phi(P_r(a)) = 1 + \sum_{(m,n) \in M_1 \cup M_2} r^{(|m|+|n|)} (h_{m,n} + \overline{h_{m,n}}) \leq 0.$$

Then let

$$r^{(|m_0|+|n_0|)} = \text{Min} \{ r^{(|m|+|n|)} : (m,n) \in M \}$$

and note that $0 \leq r^{(|m_0|+|n_0|)} \leq 1$. Now we have two cases. Either

$$-1 < \sum_{(m,n) \in M_1 \cup M_2} (h_{m,n} + \overline{h_{m,n}}) \leq 0,$$

or

$$(10) \quad \sum_{(m,n) \in M_1 \cup M_2} (h_{m,n} + \overline{h_{m,n}}) > 0.$$

In the first case, since

$$\sum_{(m,n) \in M_1 \cup M_2} (h_{m,n} + \overline{h_{m,n}}) \leq r^{(|m_0|+|n_0|)} \sum_{(m,n) \in M_1 \cup M_2} (h_{m,n} + \overline{h_{m,n}}),$$

we have

$$\begin{aligned} -1 &< \sum_{(m,n) \in M_1 \cup M_2} r^{(|m_0|+|n_0|)} (h_{m,n} + \overline{h_{m,n}}) \\ &\leq \sum_{(m,n) \in M_1 \cup M_2} r^{(|m|+|n|)} (h_{m,n} + \overline{h_{m,n}}). \end{aligned}$$

So

$$\sum_{(m,n) \in M_1 \cup M_2} r^{(|m|+|n|)} (h_{m,n} + \overline{h_{m,n}}) > -1,$$

which contradicts 9. In the second case again we have

$$\sum_{(m,n) \in M_1 \cup M_2} r^{(|m_0|+|n_0|)} (h_{m,n} + \overline{h_{m,n}})$$

$$\leq \sum_{(m,n) \in M_1 \cup M_2} r^{(|m|+|n|)} (h_{m,n} + \overline{h_{m,n}}) \leq -1,$$

which means

$$\sum_{(m,n) \in M_1 \cup M_2} r^{(|m_0|+|n_0|)} (h_{m,n} + \overline{h_{m,n}})$$

is strictly negative. But this contradicts 10, for

$$\sum_{(m,n) \in M_1 \cup M_2} r^{(|m_0|+|n_0|)} (h_{m,n} + \overline{h_{m,n}})$$

and

$$\sum_{(m,n) \in M_1 \cup M_2} (h_{m,n} + \overline{h_{m,n}})$$

have the same signs.

Then we show there exist $B_1, B_2 > 0$ such that for $0 \leq r \leq 1$, $\sigma(P_r(a)) \subseteq [B_1, B_2]$, where $\sigma(P_r(a))$ is the spectrum of $P_r(a)$. Since for $0 \leq r \leq 1$,

$$\begin{aligned} \|P_r(a)\| &= \|1 + \sum_{(m,n) \in M} a_{m,n} r^{(|m|+|n|)} U^m V^n\| \\ &\leq 1 + \sum_{(m,n) \in M} |a_{m,n}| r^{(|m|+|n|)} \|U^m V^n\| \leq 1 + \sum_{(m,n) \in M} |a_{m,n}|, \end{aligned}$$

and we know the spectral radius of $P_r(a)$ is less than $\|P_r(a)\|$, it suffices to put

$$B_2 = 1 + \sum_{(m,n) \in M} |a_{m,n}|.$$

Now suppose there is no such B_1 . So for each $n > 0$ there exists $r_n \in [0, 1]$, and $\lambda_n \in (0, \frac{1}{n})$, such that $\lambda_n \in \sigma(P_{r_n}(a))$. Obviously $\lim_{n \rightarrow \infty} \lambda_n = 0$. Since $\{r_n\}_{n=1}^{\infty}$ is a bounded sequence, it has a convergent subsequence. For simplicity we will call that sequence again $\{r_n\}_{n=1}^{\infty}$. Let $\lim_{n \rightarrow \infty} r_n = r_0$. Then $\lim_{n \rightarrow \infty} P_{r_n}(a) = P_{r_0}(a)$. Let $\text{Inv}(A_\theta)$ be the set of invertible elements in A_θ . It is an open set, hence its complement is closed. Then for $n > 0$, since $\lambda_n \in \sigma(P_{r_n}(a))$,

$$P_{r_n}(a) - \lambda_n 1 \notin \text{Inv}(A_\theta).$$

Then

$$\lim_{n \rightarrow \infty} P_{r_n}(a) - 1\lambda_n = P_{r_0}(a) \notin \text{Inv}(A_\theta),$$

which means $0 \in \sigma(P_{r_0}(a))$. But this is a contradiction, for we have shown for $0 \leq r \leq 1$, $P_r(a)$ is strictly positive.

Now we pick a neighborhood of $[B_1, B_2]$ away from the y -axis. Let

$$N = \left\{ x + iy : \frac{2B_1}{3} \leq x \leq B_2 + 1, -1 \leq y \leq 1 \right\}.$$

Clearly for $0 \leq r \leq 1$, $\sigma(P_r(a)) \subseteq N$. So by Proposition 1, for $0 \leq r \leq 1$, there exists δ_r such that for $y \in A_\theta$, $\|y - P_r(a)\| < \delta_r$ implies $\sigma(y) \subseteq N$. Since

$$P(a) : \mathbb{C} \longrightarrow A_\theta, \quad r \mapsto P_r(a)$$

is a continuous map, for δ_r there exists $\gamma_r > 0$, such that for $r' \in \mathbb{C}$, $|r' - r| \leq \gamma_r$ implies $\|P_{r'}(a) - P_r(a)\| < \delta_r$. So if $r' \in B_{\gamma_r}(r)$, then $\sigma(P_{r'}(a)) \subseteq N$ where $B_{\gamma_r}(r)$ is the 2-dimensional open ball centred at r with radius γ_r . Now let

$$W = \bigcup_{0 \leq r \leq 1} B_{\gamma_r}(r).$$

Obviously W is a complex open neighborhood of the interval $[0, 1]$ and the way that we have constructed W implies if $r \in W$, then $\sigma(P_r(a)) \subseteq N$. Since N is in the right half plane, using the standard branch of the logarithm, for $r \in W$, we can define $\log P_r(a)$. \square

3. THE MAIN RESULT

In this section we will first state our conjecture about a logarithmic Sobolev inequality on the noncommutative 2-torus and then we will prove it for certain elements.

Conjecture 3.1. *Let $a = \sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} U^m V^n$ be in A_θ^∞ and assume $a > 0$. Then*

$$(11) \quad \tau(a^2 \log a) \leq \sum_{(m,n) \in \mathbb{Z}^2} (|m| + |n|) |a_{m,n}|^2 + \|a\|_2^2 \log \|a\|_2,$$

which is the same as

$$\tau(a^2 \log a) \leq \sum_{(m,n) \in \mathbb{Z}^2} (|m| + |n|) |a_{m,n}|^2 + \tau(a^2) \log(\tau(a))^{1/2}.$$

Our main goal was of course to prove the conjecture in general using Weissler's method [15], however, because of noncommutativity, in the last step we encountered a technical problem. So we decided to restrict ourselves to a class of elements. Now we will prove the conjecture for the case $m = sn$ for some s and later on in section 4 we will give more details of what we have set up for the general case and explain what the problem is in this setting.

Theorem 3.2. Let $a = \sum_{n \in \mathbb{Z}} a_n U^n V^{sn}$ be in A_θ^∞ where $s \in \mathbb{Z} \setminus \{0\}$, such that $a > 0$. Then

$$(12) \quad \tau(a^2 \log a) \leq \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |a_n|^2 + \|a\|_2^2 \log \|a\|_2$$

which is the same as

$$\tau(a^2 \log a) \leq \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |a_n|^2 + \tau(a^2) \log(\tau(a)) \frac{1}{2}.$$

Proof. First suppose $\tau(a) = 1$ i.e. $a_0 = 1$, and suppose that at most finitely many number of a_n are nonzero. Put $x = a - 1 = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_n U^n V^{sn}$.

Using the fact that $\|a\|_2^2 = 1 + \|x\|_2^2$, it can be shown that

$$\|a\|_2^2 \log \|a\|_2 \geq \frac{1}{2} \|x\|_2^2.$$

We shall prove the theorem by proving an stronger inequality, namely

$$(13) \quad 0 \leq \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |a_n|^2 + \frac{1}{2} \|x\|_2^2 - \tau(a^2 \log a).$$

For complex number r we define

$$x_r = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} a_n r^{(1+|s|)|n|} U^n V^{sn}$$

and $P_r(a) = 1 + x_r$. By Proposition 2, for r in some complex neighborhood of the interval $[0, 1]$, we can define $\log P_r(a)$.

Let

$$\begin{aligned} G(r) &= \sum_{n \in \mathbb{Z}} (1 + |s|) r^{2(1+|s|)|n|} |n| |a_n|^2 \\ &\quad + \frac{1}{2} \|x_r\|_2^2 - \tau((P_r(a))^2 \log P_r(a)). \end{aligned}$$

Therefore, to prove (33) it suffices to show $G(1) \geq 0$. It can be shown that $G(r)$ is analytic in a complex neighborhood of $[0, 1]$. So to prove $G(1) \geq 0$, using Lemma 2.1, we shall show that all the coefficients of the expansion of $G(r)$ around $r = 0$ are nonnegative.

First note that for r with small enough $|r|$ we have $\|x_r\|_2 < 1$ (the sum is a finite sum). So

$$\begin{aligned} (P_r(a))^2 \log P_r(a) &= (1 + x_r)^2 \log(1 + x_r) \\ &= (1 + 2x_r + x_r^2) \left(1 - \frac{1}{2}x_r^2 + \frac{1}{3}x_r^3 - \frac{1}{4}x_r^4 + \cdots\right) \end{aligned}$$

$$= x_r + \frac{3}{2}x_r^2 + 2 \sum_{k=3}^{\infty} (-1)^{k-1} x_r^k \frac{(k-3)!}{k!}.$$

So

$$(14) \quad G(r) = \sum_{n \in \mathbb{Z}} (1 + |s|) r^{2(1+|s|)|n|} |n| |a_n|^2 + \frac{1}{2} \|x_r\|_2^2 \\ - \tau(x_r) - \frac{3}{2} \tau(x_r^2) + 2 \sum_{k=3}^{\infty} (-1)^k \frac{(k-3)!}{k!} \tau(x_r^k).$$

Using the facts that $\tau(x_r) = 0$ and

$$\tau(x_r^2) = \|x_r\|_2^2 = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} r^{2(1+|s|)|n|} |a_n|^2,$$

combined with (34), we get

$$(15) \quad G(r) = 2 \sum_{\substack{n \in \mathbb{Z} \\ n \geq 0}} ((1 + |s|)n - 1) r^{2(1+|s|)n} |a_n|^2 + 2 \sum_{k=3}^{\infty} g_k(r),$$

where $g_k(r) = (-1)^k \frac{(k-3)!}{k!} \tau(x_r^k)$. Now we try to find the Taylor expansion of $\tau(x_r^k)$.

First we need to fix some notations. Let

$$M = \{n \in \mathbb{Z} : n \neq 0, a_n \neq 0\},$$

$$M_1 = \{n \in M : n > 0\}.$$

For a function $P : M \rightarrow \mathbb{Z}_0^+$, we put

$$M_P = \{n \in M : P(n) \neq 0\}.$$

So (M_P, P) is a multiset. Indeed, the multiplicity of n is $P(n)$. Moreover, let $\mathcal{S}(M_P)$ be the set of all permutations of the multiset (M_P, P) . Let I_k be the set of all functions $P : M \rightarrow \mathbb{Z}_0^+$ such that

$$\sum_{n \in M} P(n) = k,$$

and $I_{k,0}$ be the set of all functions in I_k such that

$$\sum_{n \in M} P(n)n = 0.$$

For $P : M \rightarrow \mathbb{Z}_0^+$, we also define

$$Q_P : M_1 \rightarrow \mathbb{Z}_0^+$$

by $Q_P(n) = P(-n)$.

Using the multinomial expansion of x_r we have

$$x_r^k = \sum_{P \in I_k} \left(\prod_{n \in M} (a_n r^{(1+|s|)|n|})^{P(n)} \right) \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right)$$

where n_i^P , for $i = 1, 2, \dots, k$, is a labeling of elements of M_P when $P \in I_k$. Then

$$\tau(x_r^k) = \sum_{P \in I_{k,0}} \left(\prod_{n \in M} (a_n r^{(1+|s|)|n|})^{P(n)} \right) \tau \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right).$$

So we have

$$\begin{aligned} \tau(x_r^k) &= \sum_{P \in I_{k,0}} \prod_{n \in M} (a_n r^{(1+|s|)n})^{P(n)} \prod_{-n \in M_1} (a_{-n} r^{-(1+|s|)n})^{P(n)} \times \\ &\quad \tau \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right). \end{aligned}$$

And hence

$$\begin{aligned} \tau(x_r^k) &= \sum_{P \in I_{k,0}} \prod_{n \in M_1} (a_n r^{(1+|s|)n})^{P(n)} \prod_{n \in M_1} (a_{-n} r^{(1+|s|)n})^{Q_P(n)} \\ &\quad \tau \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right). \end{aligned}$$

Then since a is self-adjoint, using (4) we have

$$\begin{aligned} (16) \quad \tau(x_r^k) &= \sum_{(P,Q) \in H_k} \prod_{n \in M_1} (a_n r^{(1+|s|)n})^{P(n)} \prod_{n \in M_1} (\overline{a_n} r^{(1+|s|)n})^{Q(n)} \\ &\quad e^{(-2\pi i s \theta \sum_{n \in M_1} Q(n)n^2)} \tau \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} \prod_{i=1}^k U^{\sigma(n_i^{P,Q})} V^{s\sigma(n_i^{P,Q})} \right), \end{aligned}$$

where H_k is the set of all pairs (P, Q) such that

$$P : M_1 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \longrightarrow \mathbb{Z}_0^+,$$

$$(17) \quad \sum_{n \in M_1} P(n)n = \sum_{n \in M_1} Q(n)n,$$

$$(18) \quad \sum_{n \in M_1} P(n) + \sum_{n \in M_1} Q(n) = k.$$

Also $(M_{P,Q}, [P, Q])$ is a multiset defined by

$$M_{P,Q} = M_P^+ \cup M_Q^-,$$

where

$$M_P^+ = \{n \in M_1 : P(n) \neq 0\},$$

$$M_Q^- = \{n \in M : -n \in M_1, Q(-n) \neq 0\},$$

and

$$[P, Q] : M_{P,Q} \longrightarrow \mathbb{Z}_0^+,$$

is defined by

$$[P, Q](n) = \begin{cases} P(n) & n \in M_P^+ \\ Q(-n) & n \in M_Q^- \end{cases}.$$

Also, $n_i^{P,Q}$ for $i = 1, 2, \dots, k$ is a labeling for elements of $M_{P,Q}$.

So regarding (36), we see

$$\begin{aligned} \tau(x_r^k) &= \sum_{(P,Q) \in H_k} \prod_{n \in M_1} (a_n r^{(1+|s|)n})^{P(n)} \prod_{n \in M_1} (\overline{a}_n r^{(1+|s|)n})^{Q(n)} \\ &= e^{(-2\pi i s \theta \sum_{n \in M_1} Q(n)n^2)} \sum_{\sigma \in \mathcal{S}(M_{P,Q})} \tau \left(\prod_{i=1}^k U^{\sigma(n_i^{P,Q})} V^{s\sigma(n_i^{P,Q})} \right). \end{aligned}$$

Now we calculate $\tau \left(\prod_{i=1}^k U^{\sigma(n_i^{P,Q})} V^{s\sigma(n_i^{P,Q})} \right)$, for $\sigma \in \mathcal{S}(M_{P,Q})$, the set of permutations of the multiset $M_{P,Q}$. For simplicity we drop the superscript P, Q . Using (37), we have

$$(19) \quad \sum_{i=1}^k \sigma(n_i) = 0$$

Hence

$$\tau \left(\prod_{i=1}^k U^{\sigma(n_i)} V^{s\sigma(n_i)} \right) = \tau \left(e^{2\pi i \theta B_\sigma} U^{\sum_{i=1}^k \sigma(n_i)} V^{s \sum_{i=1}^k \sigma(n_i)} \right) = e^{2\pi i \theta B_\sigma},$$

where for $\sigma \in \mathcal{S}(M_{P,Q})$,

$$B_\sigma = \frac{s}{2} \sum_{n \in M_1} (P(n) + Q(n))n^2.$$

Infact, we know that

$$\begin{aligned} B_\sigma &= -s\sigma(n_2)\sigma(n_1) \\ &\quad -s\sigma(n_3)[\sigma(n_1) + \sigma(n_2)] \\ &\quad -s\sigma(n_4)[\sigma(n_1) + \sigma(n_2) + \sigma(n_3)] - \dots \end{aligned}$$

$$\begin{aligned}
& -s\sigma(n_{k-1}) [\sigma(n_1) + \sigma(n_2) + \cdots + \sigma(n_{k-2})] \cdot \\
& -s\sigma(n_k) [\sigma(n_1) + \sigma(n_2) + \cdots + \sigma(n_{k-1})] \cdot
\end{aligned}$$

We also define

$$\begin{aligned}
A_\sigma &= s\sigma(n_1) [\sigma(n_1) + \sigma(n_2) + \cdots + \sigma(n_k)] \\
&+ s\sigma(n_2) [\sigma(n_2) + \sigma(n_3) + \cdots + \sigma(n_k)] \\
&+ s\sigma(n_3) [\sigma(n_3) + \sigma(n_4) + \cdots + \sigma(n_k)] \\
&+ s\sigma(n_4) [\sigma(n_4) + \sigma(n_5) + \cdots + \sigma(n_k)] + \cdots \\
&+ s\sigma(n_{k-1}) [\sigma(n_{k-1}) + \sigma(n_k)] \\
&+ s\sigma(n_k)\sigma(n_k).
\end{aligned}$$

Using (19), we get $B_\sigma - A_\sigma = 0$. So $B_\sigma = \frac{1}{2}(B_\sigma + A_\sigma)$. On the other hand,

$$\begin{aligned}
B_\sigma + A_\sigma &= \sum_{i=1}^k s(\sigma(n_i))^2 = \sum_{i=1}^k s(n_i)^2 \\
&= \sum_{n \in M_P^+} P(n)sn^2 + \sum_{n \in M_Q^-} Q(-n)sn^2 = \sum_{n \in M_1} P(n)sn^2 + \sum_{-n \in M_1} Q(-n)sn^2 \\
&= \sum_{n \in M_1} P(n)sn^2 + \sum_{n \in M_1} Q(n)sn^2.
\end{aligned}$$

So we have proved

$$B_\sigma = \frac{s}{2} \sum_{n \in M_1} (P(n) + Q(n))n^2.$$

Therefore,

$$\begin{aligned}
\tau(x_r^k) &= \sum_{(P,Q) \in H_k} \prod_{n \in M_1} (a_n r^{(1+|s|)n})^{P(n)} \prod_{n \in M_1} (\overline{a_n} r^{(1+|s|)n})^{Q(n)} \\
&e^{\frac{(-2\pi i s \theta}{2} \sum_{n \in M_1} Q(n)n^2)} \sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{2\pi i \theta B_\sigma}.
\end{aligned}$$

Since

$$|\mathcal{S}(M_{P,Q})| = \frac{k!}{\prod_{n \in M_1} P(n)!Q(n)!},$$

we see that

$$\tau(x_r^k) = k! \sum_{(P,Q) \in H} \prod_{n \in M_1} (a_n r^{(1+|s|)n})^{P(n)} \prod_{n \in M_1} (\overline{a_n} r^{(1+|s|)n})^{Q(n)} \times$$

$$\begin{aligned}
& \frac{1}{\prod_{n \in M_1} P(n)!Q(n)!} e^{-2\pi i s \theta \sum_{(m,n) \in M_1} Q(n)n^2} e^{\pi i s \theta \left(\sum_{n \in M_1} (P(n)+Q(n))n^2 \right)} \\
&= k! \sum_{(P,Q) \in H} \prod_{n \in M_1} (a_n r^{(1+|s|)n})^{P(n)} \prod_{n \in M_1} (\overline{a_n} r^{(1+|s|)n})^{Q(n)} \times \\
& \quad \frac{1}{\prod_{n \in M_1} P(n)!Q(n)!} e^{\pi i s \theta \sum_{n \in M_1} (P(n)-Q(n))n^2} \\
&= k! \sum_{(P,Q) \in H} r^{\left(\sum_{n \in M_1} (1+|s|)n(P(n)+Q(n)) \right)} \times \\
& \quad e^{\pi i s \theta \sum_{n \in M_1} P(n)n^2} \prod_{n \in M_1} \frac{(a_n)^{P(n)}}{P(n)!} e^{-\pi i s \theta \sum_{n \in M_1} Q(n)n^2} \prod_{n \in M_1} \frac{(\overline{a_n})^{Q(n)}}{Q(n)!}.
\end{aligned}$$

Now for a function $P : M \longrightarrow \mathbb{Z}_0^+$, define

$$(20) \quad D(p) = e^{\pi i s \theta \sum_{n \in M_1} P(n)n^2} \prod_{n \in M_1} \frac{(-a_n)^{P(n)}}{P(n)!}.$$

Then we have

$$\tau(x_r^k) = k! \sum_{(P,Q) \in H} (-1)^{\left(\sum_{n \in M_1} P(n)+Q(n) \right)} r^{\left(\sum_{n \in M_1} (1+|s|)n(P(n)+Q(n)) \right)} D(P) \overline{D(Q)}$$

So

$$(21) \quad \tau(x_r^k) = (-1)^k k! \sum_{l=2}^{\infty} r^{2(1+|s|)l} \left(\sum_{(P,Q) \in G_l} D(P) \overline{D(Q)} \right)$$

where G_l the set of all pairs (P, Q) such that

$$P : M_1 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \longrightarrow \mathbb{Z}_0^+,$$

and

$$\begin{aligned}
& \sum_{n \in M_1} P(n) + \sum_{n \in M_1} Q(n) = k, \\
& \sum_{n \in M_1} P(n)n = \sum_{n \in M_1} Q(n)n = l,
\end{aligned}$$

One should note that in (47), l starts from 2. Here we shall show why that is the case:

$$(22) \quad \sum_{n \in M_1} P(n) \leq \sum_{n \in M_1} P(n)n.$$

Similarly we have

$$(23) \quad \sum_{n \in M_1} Q(n) \leq \sum_{n \in M_1} Q(n)n,$$

So

$$\begin{aligned} k &= \sum_{n \in M_1} P(n) + \sum_{n \in M_1} Q(n) \\ &\leq \sum_{n \in M_1} P(n)n + \sum_{n \in M_1} Q(n)n = 2l \end{aligned}$$

So for a fixed k , $\frac{k}{2} \leq l$ and since k is at least 3, $l \geq 2$.

Now for l and $t \geq 1$ define

$$C(t, l) := \sum_{P \in H_{t,l}} D(P),$$

where $H_{t,l}$ is the set of all functions $P : M_1 \rightarrow \mathbb{Z}_0^+$ such that

$$(24) \quad \sum_{n \in M_1} P(n) = t,$$

and

$$(25) \quad \sum_{n \in M_1} P(n)n = l.$$

When there is no such a P then the sum is taken to be 0. For instance if $t > l$ then there is no such a P , for

$$t = \sum_{n \in M_1} P(n) \leq \sum_{n \in M_1} P(n)n = l.$$

Then we have

$$\begin{aligned} \sum_{(P,Q) \in G_l} D(P) \overline{D(Q)} &= \sum_{t=1}^{k-1} \sum_{\substack{Q \in H_{k-t,l} \\ P \in H_{t,l}}} D(P) \overline{D(Q)} \\ &= \sum_{t=1}^{k-1} \left(\sum_{P \in H_{t,l}} D(P) \right) \left(\sum_{Q \in H_{k-t,l}} \overline{D(Q)} \right) = \sum_{t=1}^{k-1} C(t, l) \overline{C(k-t, l)}. \end{aligned}$$

Now using this in (47), we get

$$\tau(x_r^k) = (-1)^k k! \sum_{l=2}^{\infty} r^{2(1+|s|)l} \sum_{t=1}^{k-1} C(t, l) \overline{C(k-t, l)}$$

and this implies

$$\begin{aligned}
\sum_{k=3}^N g_k(r) &= \sum_{k=3}^N (k-3)! \sum_{l=2}^{\infty} r^{2(1+|s|)l} \sum_{t=1}^{k-1} C(t, l) \overline{C(k-t, l)} \\
&= \sum_{l=2}^{\infty} r^{2(1+|s|)l} \sum_{k=3}^N (k-3)! \sum_{t=1}^{k-1} C(t, l) \overline{C(k-t, l)} \\
&= \sum_{l=2}^{\infty} r^{2(1+|s|)l} \sum_{\substack{i=1 \\ i+j \geq 3}}^l \sum_{j=1}^l (i+j-3)! C(i, l) \overline{C(j, l)}.
\end{aligned}$$

Therefore, for $N \geq 2l$ the coefficient of $r^{2(1+|s|)l}$ in $\sum_{k=3}^N g_k(r)$ is

$$(26) \quad \sum_{\substack{i=1 \\ i+j \geq 3}}^l \sum_{j=1}^l (i+j-3)! C(i, l) \overline{C(j, l)},$$

and this is also the coefficient of $r^{2(1+|s|)l}$ in $\sum_{k=3}^{\infty} g_k(r)$.

Now we show for $l \geq 0$, $C(1, l) = -a_l e^{\pi i s l^2 \theta}$. Infact this is true, for if $l \notin M$, then $a_l = 0$ and also $H_{1,l} = \emptyset$ which implies $C(1, l) = 0$. If $l \in M$, then

$$P(n) = \begin{cases} 1 & n = l \\ 0 & \text{otherwise} \end{cases}$$

is the only element in $H_{1,l}$. So using (54), we have

$$(27) \quad C(1, l) = \sum_{P \in H_{1,l}} D(P) = -a_l e^{\pi i s l^2 \theta}.$$

Recall that

$$G(r) = 2 \sum_{\substack{n \in \mathbb{Z} \\ n \geq 0}} ((1+|s|)n - 1) r^{2(1+|s|)n} |a_n|^2 + 2 \sum_{k=3}^{\infty} g_k(r).$$

Therefore the coefficient of $r^{2(1+|s|)l}$, ($l \geq 2$) in $G(r)$ is

$$2((1+|s|)l - 1)|a_l|^2 + 2 \sum_{\substack{i=1 \\ i+j \geq 3}}^l \sum_{j=1}^l (i+j-3)! C(i, l) \overline{C(j, l)}.$$

Using (27), this is equal to

$$2((1+|s|)l - 1)C(1, l) \overline{C(1, l)} + 2 \sum_{\substack{i=1 \\ i+j \geq 3}}^l \sum_{j=1}^l (i+j-3)! C(i, l) \overline{C(j, l)}$$

which can be written as

$$(28) \quad 2 \sum_{i=1}^l \sum_{j=1}^l A_l(i, j) C(i, l) \overline{C(j, l)}$$

where the matrix A_l defined by

$$A_l(i, j) = \begin{cases} (1 + |s|)l - 1 & i = j = 1 \\ (i + j - 3)! & i + j \geq 3 \end{cases}.$$

In [15] it has been shown that for $l \geq 2$, A_l is a positive semi-definite matrix. Hence the coefficient of $r^{2(1+|s|)l}$, ($l \geq 2$) in $G(r)$ is positive. So we have proved (12) for a positive element a with $a_0 = 1$ in which only finitely many coefficients are non-zero.

Homogeneity of (12) implies it should hold for a positive element a (with only finitely many non-zero coefficients), even if $a_0 \neq 1$.

Finally, we shall prove (12) for an arbitrary strictly positive element of the form $\sum_{n \in \mathbb{Z}} a_n U^n V^{sn}$. For $a = \sum_{n \in \mathbb{Z}} a_n U^n V^{sn}$ and $b = \sum_{n \in \mathbb{Z}} b_n U^n V^{sn}$ in A_θ^∞ , we define

$$a * b = \sum_{p \in \mathbb{Z}} (a * b)_p U^p V^{sp}$$

where $(a * b)_p = a_p b_p$. We also define d_j in A_θ^∞ for $j \geq 0$ by

$$d_j = \sum_{n \in \mathbb{Z}} d_n^j U^n V^{sn},$$

where

$$d_n^j = \begin{cases} 1 & |n| \leq j \\ 0 & \text{otherwise} \end{cases}.$$

Then we have

$$\|(d_j * a) - a\|_2^2 = \sum_{n \in \mathbb{Z}} |d_n^j a_n - a_n|^2 = \sum_{|n| > j} |a_n|^2.$$

So

$$(29) \quad \lim_{j \rightarrow \infty} d_j * a = a,$$

in $\|\cdot\|_2$ topology.

Moreover,

$$\begin{aligned} \|(d_j * a) - a\| &= \left\| \sum_{n \in \mathbb{Z}} (d_n^j a_n - a_n) U^n V^{sn} \right\| \\ &\leq \sum_{|n| > j} |a_n|, \end{aligned}$$

which implies

$$\lim_{j \rightarrow \infty} d_j * a = a,$$

in C^* -norm topology. Now let

$$a = \sum_{n \in \mathbb{Z}} a_n U^n V^{sn}$$

be a strictly positive element in A_θ^∞ and F be a complex open neighborhood of $\sigma(a)$ in the right half plane away from the y axis. (Since a is strictly positive, we can choose such a set.) Therefore, using Proposition (1), we see for large enough j , $\sigma(d_j)$ is inside F . Hence for large enough j , we can define $\log(d_j * a)$.

Since for $j \geq 0$, $d_j * a$ is an element of A_θ^∞ which has at most finitely many non-zero coefficients, we shall apply (12) to $d_j * a$ for large enough j and we will get

$$(30) \quad \tau((d_j * a)^2 \log(d_j * a)) \leq \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |d_n^j|^2 |a_n|^2 \\ + \|d_j * a\|_2^2 \log \|d_j * a\|_2.$$

Let

$$h = \sum_{n \in \mathbb{Z}} (1 + |s|) \frac{1}{2} |n| \frac{1}{2} a_n U^n V^{sn}.$$

Then

$$\|d_j * h\|_2^2 = \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |d_n^j|^2 |a_n|^2$$

and

$$(31) \quad \lim_{j \rightarrow \infty} d_j * h = h$$

in $\|\cdot\|_2$ topology.

Thus,

$$(32) \quad \lim_{j \rightarrow \infty} \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |d_n^j|^2 |a_n|^2 = \lim_{j \rightarrow \infty} \|d_j * h\|_2^2 \\ = \lim_{j \rightarrow \infty} \|h\|_2^2 = \sum_{n \in \mathbb{Z}} (1 + |s|) |n| |a_n|^2.$$

To prove (12), taking the limit of (30) as $j \rightarrow \infty$, we use (29), (32) and also the continuity of τ with respect to $\|\cdot\|_2$. Infact, τ is continuous with respect to $\|\cdot\|_2$, for one can show (See [8] Theorem 3.3.2.) for $a \in A_\theta$,

$$|\tau(a)|^2 \leq \|\tau\| \tau(a^* a).$$

□

4. TOWARDS PROVING THE CONJECTURE

In this section, as promised in section 3, we will give the details of what we have done towards proving the Conjecture 3.1 and we will explain what the remaining technical problem is. In what follows we will use the assumptions of the Conjecture 3.1.

First suppose $\tau(a) = 1$, *i.e.* $a_{0,0} = 1$ and suppose that at most finitely many number of $a_{m,n}$ are nonzero. Put $x = a - 1 = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} a_{m,n} U^m V^n$.

Using the fact that $\|a\|_2^2 = 1 + \|x\|_2^2$, it can be shown that

$$\|a\|_2^2 \log \|a\|_2 \geq \frac{1}{2} \|x\|_2^2.$$

We are going to prove the conjecture by proving an stronger inequality:

$$(33) \quad 0 \leq \sum_{(m,n) \in \mathbb{Z}^2} (|m| + |n|) |a_{m,n}|^2 + \frac{1}{2} \|x\|_2^2 - \tau(a^2 \log a).$$

For complex number r we define

$$x_r = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} a_{m,n} r^{(|m|+|n|)} U^m V^n$$

and $P_r(a) = 1 + x_r$. By Proposition 2, for r in some complex neighborhood of the interval $[0, 1]$, we can define $\log P_r(a)$.

Let

$$\begin{aligned} G(r) &= \sum_{(m,n) \in \mathbb{Z}^2} r^{2(|m|+|n|)} (|m| + |n|) |a_{m,n}|^2 \\ &\quad + \frac{1}{2} \|x_r\|_2^2 - \tau((P_r(a))^2 \log P_r(a)). \end{aligned}$$

Therefore, to prove (33) it suffices to show that $G(1) \geq 0$. It can be shown that $G(r)$ is analytic in a complex neighborhood of $[0, 1]$. So to prove $G(1) \geq 0$, using lemma 2.1, we need to show that all the coefficients of the expansion of $G(r)$ around $r = 0$ are nonnegative.

First note that for r with small enough $|r|$ we have $\|x_r\|_2 < 1$ (the sum is a finite sum). So

$$\begin{aligned} (P_r(a))^2 \log P_r(a) &= (1 + x_r)^2 \log(1 + x_r) \\ &= (1 + 2x_r + x_r^2) \left(1 - \frac{1}{2}x_r^2 + \frac{1}{3}x_r^3 - \frac{1}{4}x_r^4 + \cdots\right) \\ &= x_r + \frac{3}{2}x_r^2 + 2 \sum_{k=3}^{\infty} (-1)^{k-1} x_r^k \frac{(k-3)!}{k!}. \end{aligned}$$

So

$$(34) \quad G(r) = \sum_{(m,n) \in \mathbb{Z}^2} r^{2(|m|+|n|)} (|m| + |n|) |a_{m,n}|^2 + \frac{1}{2} \|x_r\|_2^2 \\ - \tau(x_r) - \frac{3}{2} \tau(x_r^2) + 2 \sum_{k=3}^{\infty} (-1)^k \frac{(k-3)!}{k!} \tau(x_r^k).$$

Using the facts that $\tau(x_r) = 0$ and

$$\tau(x_r^2) = \|x_r\|_2^2 = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} r^{2(|m|+|n|)} |a_{m,n}|^2$$

in (34) we get

$$(35) \quad G(r) = 2 \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ m \geq 0, n \geq 0}} (m+n-1) r^{2(m+n)} |a_{m,n}|^2 + 2 \sum_{k=3}^{\infty} g_k(r)$$

where $g_k(r) = (-1)^k \frac{(k-3)!}{k!} \tau(x_r^k)$. Now we try to find the Taylor expansion of $\tau(x_r^k)$.

First we need to fix some notations. Let

$$M = \{(m, n) \in \mathbb{Z}^2 : (m, n) \neq 0, a_{m,n} \neq 0\},$$

$$M_1 = \{(m, n) \in M : m \geq 0, n \geq 0\},$$

$$M_2 = \{(m, n) \in M : m > 0, n < 0\}.$$

For a function $P : M \longrightarrow \mathbb{Z}_0^+$, we put

$$M_P = \{(m, n) \in M : P(m, n) \neq 0\}.$$

So (M_P, P) is a multiset. Indeed, the multiplicity of (m, n) is $P(m, n)$. Moreover, let $\mathcal{S}(M_P)$ be the set of all permutations of the multiset (M_P, P) . For $\sigma \in \mathcal{S}(M_P)$, by $\sigma_1(m, n)$ and $\sigma_2(m, n)$ we mean the first component of $\sigma(m, n)$ and the second component of that respectively. Let I_k be the set of all functions $P : M \longrightarrow \mathbb{Z}_0^+$ such that

$$\sum_{(m,n) \in M} P(m, n) = k,$$

and $I_{k,0}$ be the set of all functions in I_k such that

$$\sum_{(m,n) \in M} P(m, n) m = \sum_{(m,n) \in M} P(m, n) n = 0.$$

For $P : M \longrightarrow \mathbb{Z}_0^+$, we also define

$$Q_P : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+$$

by $Q_P(m, n) = P(-m, -n)$.

Using the multinomial expansion of x_r we have

$$x_r^k = \sum_{P \in I_k} \left(\prod_{(m,n) \in M} (a_{m,n} r^{|m|+|n|})^{P(m,n)} \right) \\ \times \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma_1(m_i^P, n_i^P)} V^{\sigma_2(m_i^P, n_i^P)} \right)$$

where (m_i^P, n_i^P) , for $i = 1, 2, \dots, k$, is a labeling of elements of M_P when $P \in I_k$. Then

$$\tau(x_r^k) = \sum_{P \in I_{k,0}} \left(\prod_{(m,n) \in M} (a_{m,n} r^{|m|+|n|})^{P(m,n)} \right) \\ \times \tau \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma_1(m_i^P, n_i^P)} V^{\sigma_2(m_i^P, n_i^P)} \right).$$

So we have

$$\tau(x_r^k) = \sum_{P \in I_{k,0}} \prod_{(m,n) \in M_1} (a_{m,n} r^{m+n})^{P(m,n)} \prod_{(-m,-n) \in M_1} (a_{m,n} r^{-m-n})^{P(m,n)} \\ \times \prod_{(m,n) \in M_2} (a_{m,n} r^{m-n})^{P(m,n)} \prod_{(-m,-n) \in M_2} (a_{m,n} r^{n-m})^{P(m,n)} \\ \times \tau \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma_1(m_i^P, n_i^P)} V^{\sigma_2(m_i^P, n_i^P)} \right)$$

Then,

$$\tau(x_r^k) = \sum_{P \in I_{k,0}} \prod_{(m,n) \in M_1} (a_{m,n} r^{m+n})^{P(m,n)} \prod_{(m,n) \in M_1} (a_{-m,-n} r^{m+n})^{Q_P(m,n)} \\ \times \prod_{(m,n) \in M_2} (a_{m,n} r^{m-n})^{P(m,n)} \prod_{(m,n) \in M_2} (a_{-m,-n} r^{m-n})^{Q_P(m,n)} \\ \times \tau \left(\sum_{\sigma \in \mathcal{S}(M_P)} \prod_{i=1}^k U^{\sigma_1(m_i^P, n_i^P)} V^{\sigma_2(m_i^P, n_i^P)} \right).$$

Then since a is self-adjoint, using (4) we have

$$(36) \quad \tau(x_r^k) = \sum_{(P,Q) \in H_k} \prod_{(m,n) \in M_1} (a_{m,n} r^{m+n})^{P(m,n)} \prod_{(m,n) \in M_1} (\overline{a_{m,n}} r^{m+n})^{Q(m,n)}$$

$$\begin{aligned}
& \times \prod_{(m,n) \in M_2} (a_{m,n} r^{m-n})^{P(m,n)} \prod_{(m,n) \in M_2} (\overline{a_{m,n}} r^{m-n})^{Q(m,n)} \\
& \times e^{(-2\pi i \theta \sum_{(m,n) \in M_1} Q(m,n)mn)} e^{(-2\pi i \theta \sum_{(m,n) \in M_2} Q(m,n)mn)} \\
& \times \tau \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} \prod_{i=1}^k U^{\sigma_1(m_i^{P,Q}, n_i^{P,Q})} V^{\sigma_2(m_i^{P,Q}, n_i^{P,Q})} \right),
\end{aligned}$$

where H_k is the set of all pairs (P, Q) such that

$$\begin{aligned}
P &: M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+, \\
Q &: M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+, \\
(37) \quad \sum_{(m,n) \in M_1 \cup M_2} P(m,n)n &= \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)n,
\end{aligned}$$

$$(38) \quad \sum_{(m,n) \in M_1 \cup M_2} P(m,n)m = \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)m,$$

$$(39) \quad \sum_{(m,n) \in M_1 \cup M_2} P(m,n) + \sum_{(m,n) \in M_1 \cup M_2} Q(m,n) = k$$

and $(M_{P,Q}, [P, Q])$ is a multiset defined by

$$M_{P,Q} = M_P^{1,2} \cup M_Q^{-1,-2}$$

where

$$M_P^{1,2} = \{(m,n) \in M_1 \cup M_2 : P(m,n) \neq 0\},$$

$$M_Q^{-1,-2} = \{(m,n) \in M : (m,n) \notin M_1 \cup M_2, Q(-m, -n) \neq 0\}$$

and

$$[P, Q] : M_{P,Q} \longrightarrow \mathbb{Z}_0^+,$$

is defined by

$$[P, Q](m, n) = \begin{cases} P(m, n) & (m, n) \in M_P^{1,2} \\ Q(-m, -n) & (m, n) \in M_Q^{-1,-2} \end{cases}$$

Also, $(m_i^{P,Q}, n_i^{P,Q})$ for $i = 1, 2, \dots, k$ is a labeling for elements of $M_{P,Q}$.

Now we show that for $(P, Q) \in H_k$,

$$(40) \quad \tau \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} \prod_{i=1}^k U^{\sigma_1(m_i^{P,Q}, n_i^{P,Q})} V^{\sigma_2(m_i^{P,Q}, n_i^{P,Q})} \right)$$

$$= e^{\sum_{(m,n) \in M_1 \cup M_2} \pi i \theta (P(m,n) + Q(m,n)) mn} B_{P,Q},$$

where $B_{P,Q}$ is a real number which depends on P and Q . Indeed, for $(P, Q) \in H_k$ and $\sigma \in \mathcal{S}(M_{P,Q})$ we have (for simplicity we drop the superscript P, Q)

$$(41) \quad \prod_{i=1}^k U^{\sigma_1(m_i, n_i)} V^{\sigma_2(m_i, n_i)} = e^{2\pi i \theta B_\sigma} U^{(\sum_{i=1}^k \sigma_1(m_i, n_i))} V^{(\sum_{i=1}^k \sigma_2(m_i, n_i))},$$

where

$$\begin{aligned} B_\sigma &= -\sigma_1(m_2, n_2) \sigma_2(m_1, n_1) \\ &\quad -\sigma_1(m_3, n_3) [\sigma_2(m_1, n_1) + \sigma_2(m_2, n_2)] \\ &\quad -\sigma_1(m_4, n_4) [\sigma_2(m_1, n_1) + \sigma_2(m_2, n_2) + \sigma_2(m_3, n_3)] - \cdots \\ &\quad -\sigma_1(m_k, n_k) [\sigma_2(m_1, n_1) + \sigma_2(m_2, n_2) + \sigma_2(m_3, n_3) + \cdots + \sigma_2(m_{k-1}, n_{k-1})]. \end{aligned}$$

Since $(P, Q) \in H_k$, (37) and (38) implies

$$(42) \quad \sum_{i=1}^k \sigma_1(m_i, n_i) = 0$$

and

$$(43) \quad \sum_{i=1}^k \sigma_2(m_i, n_i) = 0.$$

So using (41),

$$(44) \quad \tau\left(\prod_{i=1}^k U^{\sigma_1(m_i, n_i)} V^{\sigma_2(m_i, n_i)}\right) = e^{2\pi i \theta B_\sigma}.$$

Let

$$\begin{aligned} A_\sigma &= \sigma_1(m_1, n_1) [\sigma_2(m_1, n_1) + \sigma_2(m_2, n_2) + \cdots + \sigma_2(m_k, n_k)] \\ &\quad + \sigma_1(m_2, n_2) [\sigma_2(m_2, n_2) + \sigma_2(m_3, n_3) + \cdots + \sigma_2(m_k, n_k)] \\ &\quad + \sigma_1(m_3, n_3) [\sigma_2(m_3, n_3) + \sigma_2(m_4, n_4) + \cdots + \sigma_2(m_k, n_k)] + \cdots \\ &\quad + \sigma_1(m_{k-1}, n_{k-1}) [\sigma_2(m_{k-1}, n_{k-1}) + \sigma_2(m_k, n_k)] \\ &\quad + \sigma_1(m_k, n_k) \sigma_2(m_k, n_k). \end{aligned}$$

Using (42) and (43), one can easily check that $B_\sigma - A_\sigma = 0$. So

$$(45) \quad B_\sigma = \frac{1}{2}(B_\sigma + A_\sigma).$$

We also set

$$D_\sigma = \sum_{j=2}^k \sum_{i=1}^{j-1} [\sigma_1(m_i, n_i) \sigma_2(m_j, n_j) - \sigma_1(m_j, n_j) \sigma_2(m_i, n_i)].$$

One can easily see that

$$(46) \quad D_\sigma = \sum_{j=1}^{k-1} \sum_{i=j+1}^k [\sigma_1(m_j, n_j) \sigma_2(m_i, n_i) - \sigma_1(m_i, n_i) \sigma_2(m_j, n_j)].$$

Then we see that

$$\begin{aligned} B_\sigma + A_\sigma &= D_\sigma + \sum_{i=1}^k \sigma_1(m_i, n_i) \sigma_2(m_i, n_i) \\ &= D_\sigma + \sum_{(m,n) \in M_{P,Q}} [P, Q](m, n) mn \\ &= D_\sigma + \sum_{(m,n) \in M_{P,Q}^{1,2}} P(m, n) mn + \sum_{(m,n) \in M_{P,Q}^{-1,-2}} Q(-m, -n) mn \\ &= D_\sigma + \sum_{(m,n) \in M_1 \cup M_2} P(m, n) mn + \sum_{(-m,-n) \in M_1 \cup M_2} Q(-m, -n) mn \\ &= D_\sigma + \sum_{(m,n) \in M_1 \cup M_2} P(m, n) mn + \sum_{(m,n) \in M_1 \cup M_2} Q(m, n) mn \end{aligned}$$

Therefore, by (45) we have

$$B_\sigma = \frac{1}{2} \left[D_\sigma + \sum_{(m,n) \in M_1 \cup M_2} (P(m, n) + Q(m, n)) mn \right].$$

Then regarding (44) we have

$$\begin{aligned} \tau \left(\prod_{i=1}^k U^{\sigma_1(m_i, n_i)} V^{\sigma_2(m_i, n_i)} \right) &= e^{2\pi i \theta B_\sigma} \\ &= e^{\frac{\pi i \theta}{2} \sum_{(m,n) \in M_1 \cup M_2} (P(m,n) + Q(m,n)) mn} e^{\pi i \theta D_\sigma} \end{aligned}$$

Now if we define

$$B_{P,Q} = \sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_\sigma},$$

we see that

$$\tau \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} \prod_{i=1}^k U^{\sigma_1(m_i^{P,Q}, n_i^{P,Q})} V^{\sigma_2(m_i^{P,Q}, n_i^{P,Q})} \right).$$

$$= e^{\sum_{(m,n) \in M_1 \cup M_2} \pi i \theta (P(m,n) + Q(m,n)) mn} B_{P,Q}.$$

So we have proved (40). Now we will show that $B_{P,Q}$ is a real number. Indeed, for $\sigma \in \mathcal{S}(M_{P,Q})$, we define $\beta_\sigma \in \mathcal{S}(M_{P,Q})$ by

$$\beta_\sigma(m_i, n_i) = \sigma(m_{k-i+1}, n_{k-i+1}) \quad i = 1, 2, \dots, k.$$

Then we have

$$\begin{aligned} D_{\beta_\sigma} &= \sum_{j=2}^k \sum_{i=1}^{j-1} [\beta_{\sigma_1}(m_i, n_i) \beta_{\sigma_2}(m_j, n_j) - \beta_{\sigma_1}(m_j, n_j) \beta_{\sigma_2}(m_i, n_i)] \\ &= \sum_{j=2}^k \sum_{i=1}^{j-1} \sigma_1(m_{k-i+1}, n_{k-i+1}) \sigma_2(m_{k-j+1}, n_{k-j+1}) \\ &\quad - \sum_{j=2}^k \sum_{i=1}^{j-1} \sigma_1(m_{k-j+1}, n_{k-j+1}) \sigma_2(m_{k-i+1}, n_{k-i+1}) \\ &= \sum_{t=k-1}^1 \sum_{s=k}^{t+1} [\sigma_1(m_s, n_s) \sigma_2(m_t, n_t) - \sigma_1(m_t, n_t) \sigma_2(m_s, n_s)] \\ &= \sum_{t=1}^{k-1} \sum_{s=t+1}^k [\sigma_1(m_s, n_s) \sigma_2(m_t, n_t) - \sigma_1(m_t, n_t) \sigma_2(m_s, n_s)] \\ &= - \sum_{t=1}^{k-1} \sum_{s=t+1}^k [\sigma_1(m_t, n_t) \sigma_2(m_s, n_s) - \sigma_1(m_s, n_s) \sigma_2(m_t, n_t)] = -D_\sigma, \end{aligned}$$

where in the last equality we have used (46). Now we have

$$\begin{aligned} B_{P,Q} &= \sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_\sigma} = \frac{1}{2} \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_\sigma} + \sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_{\beta_\sigma}} \right) \\ &= \frac{1}{2} \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_\sigma} + \sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_{\beta_\sigma}} \right) \\ &= \frac{1}{2} \left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{\pi i \theta D_\sigma} + \sum_{\sigma \in \mathcal{S}(M_{P,Q})} e^{-\pi i \theta D_\sigma} \right) = \frac{1}{2} \sum_{\sigma \in \mathcal{S}(M_{P,Q})} (e^{\pi i \theta D_\sigma} + e^{-\pi i \theta D_\sigma}) \end{aligned}$$

So $B_{P,Q}$ is real.

Now using (36), we see

$$\begin{aligned} \tau(x_r^k) &= \sum_{(P,Q) \in H_k} \prod_{(m,n) \in M_1} (a_{m,n} r^{m+n})^{P(m,n)} \prod_{(m,n) \in M_1} (\overline{a_{m,n}} r^{m+n})^{Q(m,n)} \\ &\quad \prod_{(m,n) \in M_2} (a_{m,n} r^{m-n})^{P(m,n)} \prod_{(m,n) \in M_2} (\overline{a_{m,n}} r^{m-n})^{Q(m,n)} \end{aligned}$$

$$e^{-2\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)mn} e^{\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} (P(m,n)+Q(m,n))mn} B_{P,Q}$$

Then we have

$$\begin{aligned} \tau(x_r^k) &= \sum_{(P,Q) \in H_k} r^{\left(\sum_{(m,n) \in M_1} (P(m,n)+Q(m,n))(m+n) + \sum_{(m,n) \in M_2} (P(m,n)+Q(m,n))(m-n) \right)} \\ &\quad e^{\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} P(m,n)mn} e^{-\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)mn} \\ &\quad \prod_{(m,n) \in M_1 \cup M_2} a_{m,n}^{P(m,n)} \prod_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}}^{Q(m,n)} B_{P,Q} \\ &= \sum_{(P,Q) \in H_k} r^{\left(\sum_{(m,n) \in M_1 \cup M_2} P(m,n)m + \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)m \right)} \\ &\quad r^{\left(\sum_{(m,n) \in M_1} P(m,n)n - \sum_{(m,n) \in M_2} Q(m,n)n + \sum_{(m,n) \in M_1} Q(m,n)n - \sum_{(m,n) \in M_2} P(m,n)n \right)} \\ &\quad e^{\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} P(m,n)mn} e^{-\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)mn} \\ &\quad \prod_{(m,n) \in M_1 \cup M_2} a_{m,n}^{P(m,n)} \prod_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}}^{Q(m,n)} B_{P,Q} \end{aligned}$$

So

$$(47) \quad \tau(x_r^k) = \sum_{\substack{l+s=2 \\ l \geq 0, s \geq 0}}^{\infty} r^{2(l+s)} \sum_{(P,Q) \in G_{l,s}} e^{\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} P(m,n)mn} \\ \times e^{-\pi i\theta \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)mn} \prod_{(m,n) \in M_1 \cup M_2} a_{m,n}^{P(m,n)} \prod_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}}^{Q(m,n)} B_{P,Q}$$

where $G_{l,s}$ is the set of all pairs (P, Q) such that

$$P : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+,$$

and

$$\begin{aligned} &\sum_{(m,n) \in M_1 \cup M_2} P(m,n) + \sum_{(m,n) \in M_1 \cup M_2} Q(m,n) = k, \\ &\sum_{(m,n) \in M_1 \cup M_2} P(m,n)m = \sum_{(m,n) \in M_1 \cup M_2} Q(m,n)m = l, \\ &\sum_{(m,n) \in M_1} P(m,n)n - \sum_{(m,n) \in M_2} Q(m,n)n \\ &= \sum_{(m,n) \in M_1} Q(m,n)n - \sum_{(m,n) \in M_2} P(m,n)n = s. \end{aligned}$$

One should note that in (47) $(l + s)$ starts from 2. Here we shall show why that is the case: for $(m, n) \in M_1$ if $m = 0$, then $n \neq 0$. So we have

$$(48) \quad \sum_{(m,n) \in M_1} P(m, n) \leq \sum_{(m,n) \in M_1} P(m, n)m + \sum_{(m,n) \in M_1} P(m, n)n.$$

Similarly we have

$$(49) \quad \sum_{(m,n) \in M_2} P(m, n) \leq \sum_{(m,n) \in M_2} P(m, n)m - \sum_{(m,n) \in M_2} P(m, n)n,$$

$$(50) \quad \sum_{(m,n) \in M_1} Q(m, n) \leq \sum_{(m,n) \in M_1} Q(m, n)m + \sum_{(m,n) \in M_1} Q(m, n)n,$$

$$(51) \quad \sum_{(m,n) \in M_2} Q(m, n) \leq \sum_{(m,n) \in M_2} Q(m, n)m - \sum_{(m,n) \in M_2} Q(m, n)n.$$

So

$$\begin{aligned} k &= \sum_{(m,n) \in M_1 \cup M_2} P(m, n) + \sum_{(m,n) \in M_1 \cup M_2} Q(m, n) \leq \\ &\quad \sum_{(m,n) \in M_1 \cup M_2} P(m, n)m + \sum_{(m,n) \in M_1 \cup M_2} Q(m, n)m \\ &\quad + \sum_{(m,n) \in M_1} P(m, n)n - \sum_{(m,n) \in M_2} Q(m, n)n \\ &\quad + \sum_{(m,n) \in M_1} Q(m, n)n - \sum_{(m,n) \in M_2} P(m, n)n \\ &= 2(s + l) \end{aligned}$$

So for a fixed k , $\frac{k}{2} \leq l + s$ and since k is at least 3, $l + s \geq 2$.

Let $\tilde{G}_{l,s}$ be the set of all pairs (\tilde{P}, \tilde{Q}) such that

$$\tilde{P} : M \longrightarrow \mathbb{Z}_0^+,$$

$$\tilde{Q} : M \longrightarrow \mathbb{Z}_0^+,$$

and

$$\begin{aligned} \sum_{(m,n) \in M_1 \cup M_2} \tilde{P}(m, n) + \sum_{(m,n) \in M_1 \cup M_2} \tilde{Q}(m, n) &= k, \\ \sum_{(m,n) \in M_1 \cup M_2} \tilde{P}(m, n)m &= \sum_{(m,n) \in M_1 \cup M_2} \tilde{Q}(m, n)m = l, \\ \sum_{(m,n) \in M_1} \tilde{P}(m, n)n - \sum_{(m,n) \in M_2} \tilde{Q}(m, n)n &= 0 \end{aligned}$$

$$= \sum_{(m,n) \in M_1} \tilde{Q}(m,n)n - \sum_{(m,n) \in M_2} \tilde{P}(m,n)n = s.$$

There exist a one to one correspondence between $\tilde{G}_{l,s}$ and $G_{l,s}$. Infact, for $(P, Q) \in G_{l,s}$, we can define

$$\tilde{P}(m,n) = \begin{cases} P(m,n) & (m,n) \in M_1 \cup M_2 \\ Q(-m,-n) & (m,n) \notin M_1 \cup M_2 \end{cases}$$

and

$$\tilde{Q}(m,n) = \begin{cases} Q(m,n) & (m,n) \in M_1 \cup M_2 \\ P(-m,-n) & (m,n) \notin M_1 \cup M_2 \end{cases}$$

Using this correspondence and the fact that $B_{P,Q} = B_{\tilde{P}|_{M_1 \cup M_2}, \tilde{Q}|_{M_1 \cup M_2}}$ in (47), we have

$$(52) \quad \tau(x_r^k) = \sum_{\substack{l+s=2 \\ l \geq 0, s \geq 0}}^{\infty} r^{2(l+s)} \sum_{(\tilde{P}, \tilde{Q}) \in \tilde{G}_{l,s}} e^{\pi i \theta \sum_{(m,n) \in M_1 \cup M_2} \tilde{P}(m,n)mn} \\ \times e^{-\pi i \theta \sum_{(m,n) \in M_1 \cup M_2} \tilde{Q}(m,n)mn} \prod_{(m,n) \in M_1 \cup M_2} a_{m,n}^{\tilde{P}(m,n)} \prod_{(m,n) \in M_1 \cup M_2} \overline{a_{m,n}}^{\tilde{Q}(m,n)} B_{\tilde{P}, \tilde{Q}}.$$

Now If we can decompose $B_{\tilde{P}, \tilde{Q}}$ into two terms $B_{\tilde{P}}$ and $B_{\tilde{Q}}$, i.e.

$$(53) \quad B_{\tilde{P}, \tilde{Q}} = B_{\tilde{P}} B_{\tilde{Q}}$$

such that $B_{\tilde{P}}$ and $B_{\tilde{Q}}$ depend only respectively on \tilde{P} and \tilde{Q} , then we can easily continue the proof of Theorem 3.2. Indeed, If (53) holds, for a function

$$P : M \longrightarrow \mathbb{Z}_0^+,$$

we can define

$$(54) \quad D(P) = e^{\pi i \theta \sum_{(m,n) \in M_1 \cup M_2} P(m,n)mn} \prod_{(m,n) \in M_1 \cup M_2} (-a_{m,n})^{P(m,n)} B_P,$$

and the rest would be much similar to the proof of Theorem 3.2.

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